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One-dimensional shock turbulence in a compressible fluid

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The interactions of weak nonlinear disturbances in a compressible fluid including shocks, expansion waves and contact surfaces are investigated by making use of the reductive perturbation method. It is found that the nonlinear waves belonging to different families of characteristics behave almost independently of each other, while those belonging to the same family are governed by either the Burgers equation or the equation of heat conduction. Thus the statistical properties of one-dimensional shock turbulence in a compressible fluid are reduced to those of the solutions of the Burgers equation. In particular, the law of energy decay of shock turbulence is shown to be identical to that of Burgers turbulence.

1. Introduction

Turbulence in an incompressible fluid is composed of random shear motions and its velocity field, when Fourier analysed, has only transverse components in the wavenumber vector. In a compressible fluid, on the other hand, the density fluctuates together with the velocity and the pressure; and the turbulent field includes longitudinal components as well, which, unlike the transverse components, propagate in all directions as sound waves. The transverse components are well known as *shear turbulence*, and the longitudinal components may be called *compression turbulence*.

In a compressible fluid shear turbulence always causes density fluctuation or compression turbulence; and conversely the interaction of compression waves produces vorticity fluctuation or shear turbulence. At small values of the Mach number $M = U_0/a_0$ (U_0 being the characteristic velocity of turbulence, and a_0 the sound velocity of undisturbed fluid), turbulence is usually excited by the instability of laminar flows; therefore the energy is supplied primarily to shear turbulence, then transferred to compression turbulence. The situation is different for large values of M; for example, in the interstellar gas, the random heating of the gas by stars gives rise to compression turbulence, and shear turbulence is produced secondarily (see Burgers 1955*b*). Thus, in turbulence in a compressible fluid, there is continual exchange of energy between the longitudinal and transverse components of turbulence; and in equilibrium we may expect an equipartition of energy between these two components of turbulence.

The spectral representation of homogeneous turbulence in a compressible fluid was first studied by Moyal (1952). It was shown that, for turbulence at small M, the dynamical equations break up into two distinct groups, one referring to the transverse velocity spectrum, the other to the longitudinal spectra, including the longitudinal velocity components, the density and the temperature. While the spectra belonging to the latter group are strongly coupled to each other by the linear terms of the equations, the interaction between the two groups is exclusively due to the nonlinear terms, and thus becomes small at small values of M.

The properties of sound generated by shear turbulence of a given character were investigated by Lighthill (1952, 1954), who showed that the total acoustic power output is roughly proportional to the eighth power of U_0 for shear turbulence, and that the efficiency of conversion of energy from shear to compression turbulence is proportional to the fifth power of M for shear turbulence.

The foregoing analyses show conclusively that the interaction between the transverse and longitudinal components of turbulence is very weak at small M. On the other hand, the structure of the compression turbulence itself is much more sensitive to M. So long as M is infinitesimally small, the compression turbulence takes the form of a random assembly of ordinary sound waves. At M of finite magnitude comparable with 1 and even larger, however, the compression sides of sound waves rapidly steepen and become shock fronts, while the expansion sides become less and less steep; thus sound waves develop into trains of triangular shock waves or 'N-waves'. At this stage, the compression turbulence may be imagined as a random assembly of shock waves of all shapes, rushing about in all directions, with regions of gradual expansion between them (see Lighthill 1955).

In this stage of turbulence, the viscous dissipation takes place almost entirely inside the shock fronts where the velocity gradient is extremely large, and the decay of kinetic energy is largely accelerated by the presence of shock waves. Another important feature of this shock turbulence is the coalescence of shock fronts, which occurs whenever a shock front is overtaken by another. It is known that the coalescence has the effect of delaying the decay of turbulent energy (see Tatsumi & Kida 1972). Thus the statistical properties of shock turbulence may be expected to be substantially different from those of random sound waves at infinitesimal Mach numbers.

We shall investigate in the present paper the statistical mechanics of compression turbulence in isolation, taking one-dimensional motions as the immediate subject of study. The properties of one-dimensional nonlinear waves in a compressible fluid have been investigated for many years by several authors; now the individual behaviour of shock waves, expansion waves and contact surfaces, and various types of interaction between them, seem to have become familiar (see e.g. Courant & Friedrichs 1948). To deal with one-dimensional compression turbulence, or a random assembly of the above nonlinear waves, however, it is necessary to know the laws of the interactions more systematically, and, if possible, to deduce simple features of them, so as to make statistical treatment easier.

The interactions between weak shock waves and expansion waves were discussed by Burgers (1955*a*), using a modified Burgers equation and the equation of continuity, and some simple relations of weak nonlinear waves were obtained. In the present paper, we shall deal with a set of exact equations governing one-dimensional motions in a compressible fluid, and derive all the information about the weak nonlinear waves and their interactions, by applying a small parameter expansion to the above set of equations.

In § 2 various types of interaction between three elementary modes of disturbance in a compressible fluid (namely shock waves, expansion waves and contact surfaces) are surveyed in brief; and the interactions are examined in more detail for the case of weak disturbances.

In § 3 the interactions of weak nonlinear waves in a compressible, viscous and heat-conducting fluid are investigated in general, by making use of the reductive perturbation method developed by Taniuti & Wei (1968), and the procedure employed by Oikawa & Yajima (1973) for dealing with similar problems. It is shown that the nonlinear waves which belong to different families of characteristics behave almost independently of each other, and are governed separately by either the Burgers equation or the equation of heat conduction.

As examples of the analysis in § 3, the interactions of shocks and expansion waves are dealt with in § 4. In the case of head-on collision, a pair of the waves penetrate each other, suffering no change in their strengths, the only effect of collision being a change in the phase velocity of each wave by an amount proportional to the strength of the other wave. In the case of the overtaking of one wave by another, on the other hand, the behaviour of the waves is completely governed by the Burgers equation; and the coalescence of shocks and expansion waves follows as a property of solutions of the Burgers equation.

As an important corollary of the results of § 3, it follows that the asymptotic form of a weak nonlinear disturbance of an arbitrary initial form for very large Reynolds numbers and time is expressed as the sum of the asymptotic forms of solutions of the Burgers equation. It is shown in § 5 that the statistical properties of weak shock turbulence in a compressible fluid are in fact reducible to those of Burgers turbulence, which was already dealt with by Tatsumi & Kida (1972). In particular, the law of decay of turbulent energy is shown to be identical to that of Burgers turbulence.

2. Interaction of shocks, expansion waves and contact surfaces

The dynamical disturbances that take place in a uniform compressible fluid consist of three elementary modes: shock waves, expansion waves and contact surfaces. These waves, obeying nonlinear equations, interact with each other in various manners (Courant & Friedrichs 1948). Amongst all possible interactions of these waves, some typical ones are shown graphically in figure 1 (where x denotes the co-ordinate and t the time, and thick solid lines represent shock waves,

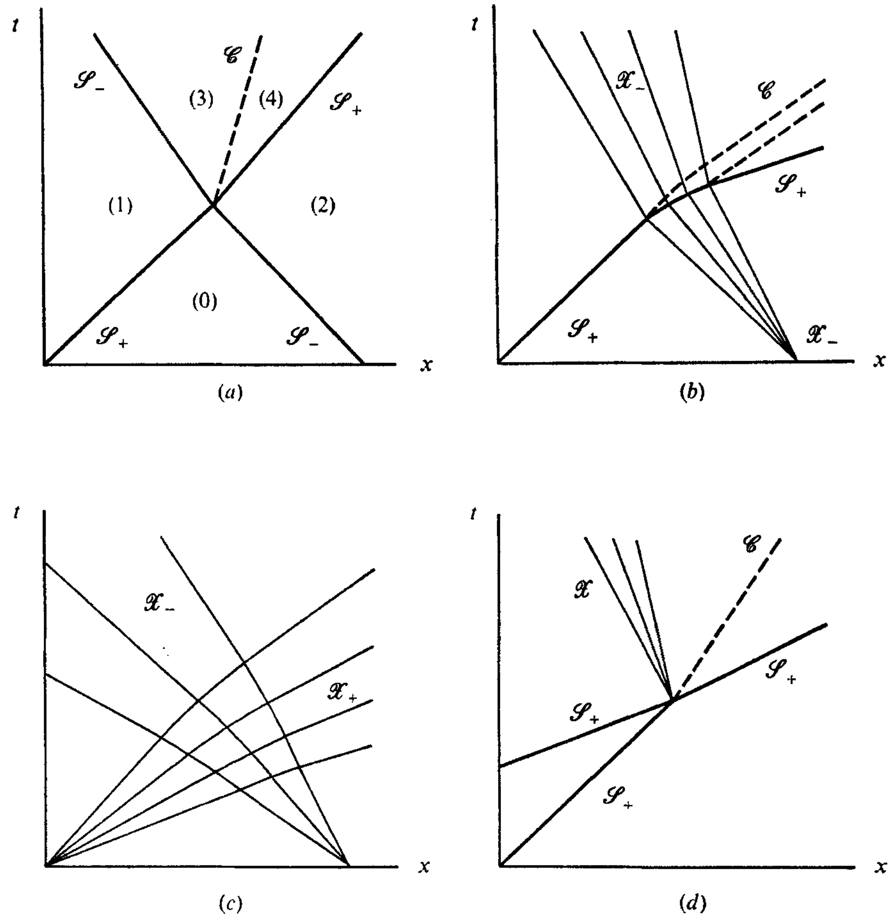

FIGURE 1. Interaction of shocks and expansion waves: ▬▬, shocks; ——, expansion waves; – – –, contact surfaces.

thin solid lines expansion waves and broken lines contact surfaces). Shock waves and expansion waves are classified into two groups: forward-facing waves, which propagate in the direction of increasing x relative to the upstream fluid, and backward-facing waves, propagating oppositely. Let us denote the forward-facing shocks and expansion waves by $\mathscr{S}_+$ and $\mathscr{X}_+$, the backward-facing shocks and expansion waves by $\mathscr{S}_-$ and $\mathscr{X}_-$, respectively.

Figures 1(a)–(c) show the head-on collision of two oppositely facing waves, a pair of shocks $\mathscr{S}_+$ and $\mathscr{S}_-$, a shock $\mathscr{S}_+$ and an expansion wave $\mathscr{X}_-$, and a pair of expansion waves $\mathscr{X}_+$ and $\mathscr{X}_-$, respectively, whereas figure 1(d) shows the overtaking of a shock $\mathscr{S}_+$ by another similarly facing shock $\mathscr{S}_+$. As may be observed from these figures, the interactions between shocks and expansion waves generally produce a contact surface $\mathscr{C}$, in addition to new shocks and expansion waves. Thus, to describe the disturbances in a compressible fluid in general, all three of the elementary nonlinear waves are required.

The situation is considerably simplified if we restrict ourselves to the case where all nonlinear waves have finite but small amplitudes. For instance, let us examine the head-on collision of two shocks. Before the shocks meet, we have three regions, say 1, 0 and 2, in the order from left to right (see figure 1 (*a*)); after collision, there appear in general two new regions between the regions 1 and 2, say 3 and 4, which are separated by a contact surface.

The density ρ, the pressure p, the fluid velocity u and other variables in the region, i say, ahead of a shock and the region, j say, behind the shock, are connected to each other by the laws of conservation of mass, momentum and energy:

$$\left.\begin{aligned} \rho_i(u_i - U) &= \rho_j(u_j - U), \\ \rho_i(u_i - U)^2 + p_i &= \rho_j(u_j - U)^2 + p_j, \\ \tfrac{1}{2}(u_i - U)^2 + E_i + \frac{p_i}{\rho_i} &= \tfrac{1}{2}(u_j - U)^2 + E_j + \frac{p_j}{\rho_j}, \end{aligned}\right\} \tag{2.1}$$

where U denotes the phase velocity of the shock, E the internal energy of fluid per unit mass, and suffixes refer to the numbers of the regions. The fluid is assumed to be an ideal polytropic gas with the equation of state

$$p = \mathscr{R}\rho T, \quad \mathscr{R} = c_p - c_V, \tag{2.2}$$

and internal energy

$$E = c_V T = \frac{1}{\gamma - 1}\frac{p}{\rho}, \quad \gamma = \frac{c_p}{c_V}, \tag{2.3}$$

where T is the temperature, $\mathscr{R}$ the gas constant, c_p and c_V the specific heats at constant pressure and constant volume respectively. Manipulation of (2.1), (2.2) and (2.3) gives the shock relations

$$\frac{\rho_j}{\rho_i} = \frac{(\gamma+1)p_j + (\gamma-1)p_i}{(\gamma+1)p_i + (\gamma-1)p_j}, \tag{2.4}$$

$$u_j - u_i = \pm(p_j - p_i)\left(\frac{2}{\rho_i[(\gamma+1)p_j + (\gamma-1)p_i]}\right)^{\frac{1}{2}}, \tag{2.5}$$

where plus and minus signs apply for the forward- and backward-facing shocks, respectively, and

$$U = \tfrac{1}{2}(u_i + u_j) + \frac{1}{\gamma - 1}\frac{a_j^2 - a_i^2}{u_j - u_i}, \tag{2.6}$$

where

$$a = (\gamma p/\rho)^{\frac{1}{2}} = (\gamma \mathscr{R} T)^{\frac{1}{2}} \tag{2.7}$$

denotes the local sound velocity.

Since the regions 3 and 4 are separated by a contact surface, it follows by definition that

$$p_3 = p_4, \quad u_3 = u_4. \tag{2.8}$$

If the shock waves are of small amplitude,

$$p_j/p_i = 1 + p_{ij}, \tag{2.9}$$

and we may neglect terms $O(|p_{ij}|^2)$. Then, substitution of (2.4) and (2.9) into (2.5) gives

$$p_{24} = p_{01}, \quad p_{13} = p_{02}, \tag{2.10}$$

which shows that, in the weak-shock approximation, the shocks pass through each other, suffering no change in their strengths.

Next, applying the same approximation to (2.4), we obtain

$$\frac{\rho_j}{\rho_i} = 1 + \frac{1}{\gamma} p_{ij}, \tag{2.11}$$

and hence

$$\frac{\rho_4}{\rho_3} = 1 + \frac{1}{\gamma}(p_{24} + p_{02} - p_{13} - p_{01}) = 1, \tag{2.12}$$

thanks to (2.10), so that no contact surface exists between regions 3 and 4. Thus, in the weak-shock approximation, the head-on collision of shocks produces only another pair of shocks of unchanged strengths, as depicted in figure 2(*a*).

For weak shocks, it follows from (2.5) that

$$u_j - u_i = \pm \frac{a_i}{\gamma} p_{ij}; \tag{2.13}$$

and, from (2.9) and (2.11), it follows that

$$\begin{aligned} a_j - a_i &= \left(\left(\frac{p_j \rho_j}{p_i \rho_i}\right)^{\frac{1}{2}} - 1\right) a_i = \frac{\gamma - 1}{2\gamma} a_i p_{ij} \\ &= \pm \tfrac{1}{2}(\gamma - 1)(u_j - u_i). \end{aligned} \tag{2.14}$$

Substitution of (2.14) into (2.6) immediately gives

$$U = \tfrac{1}{2}(u_i \pm a_i + u_j \pm a_j), \tag{2.15}$$

so that the phase velocity U of a weak shock is given by the average of the velocities of the sound waves $u_i \pm a_i$ ahead of the shock front, and $u_j \pm a_j$ behind it.

Thus, the velocities of the shocks $\mathscr{S}_+$ and $\mathscr{S}_-$ in figure 2(*a*) are expressed as follows (where the region 0 has been taken to be in undisturbed state $u_0 = 0$). Before collision,

$$\left.\begin{aligned} U_+ &= a_0 + \tfrac{1}{4}(\gamma + 1) u_1, \\ U_- &= -a_0 + \tfrac{1}{4}(\gamma + 1) u_2. \end{aligned}\right\} \tag{2.16}$$

After collision,

$$\left.\begin{aligned} U_+ &= a_0 + \tfrac{1}{4}(\gamma + 1) u_1 + \tfrac{1}{2}(3 - \gamma) u_2, \\ U_- &= -a_0 + \tfrac{1}{4}(\gamma + 1) u_2 + \tfrac{1}{2}(3 - \gamma) u_1, \end{aligned}\right\} \tag{2.17}$$

where the relations (2.8), (2.10), (2.13) and (2.14) have been taken into account.

The other types of interaction corresponding to figures 1(*b*)–(*d*) are also simplified when the waves have finite but small amplitudes. The simplified patterns of interaction are shown graphically in figures 2(*b*)–(*d*), from which it may be observed that no contact surface arises from interactions.

It may be seen, in figure 2, that there is a substantial difference between the consequences of the head-on collision of two oppositely-facing waves, and those of the overtaking of two like-facing waves. In case of head-on collision (figures 2(*a*)–(*c*)), the two waves inter-penetrate, their strengths remaining unchanged. In overtaking (figures 2(*d*)), on the other hand, two shocks coalesce to make a single shock. Similarly, it may be shown that the overtaking of a shock or an expansion wave by another expansion wave or a shock, respectively, pro-

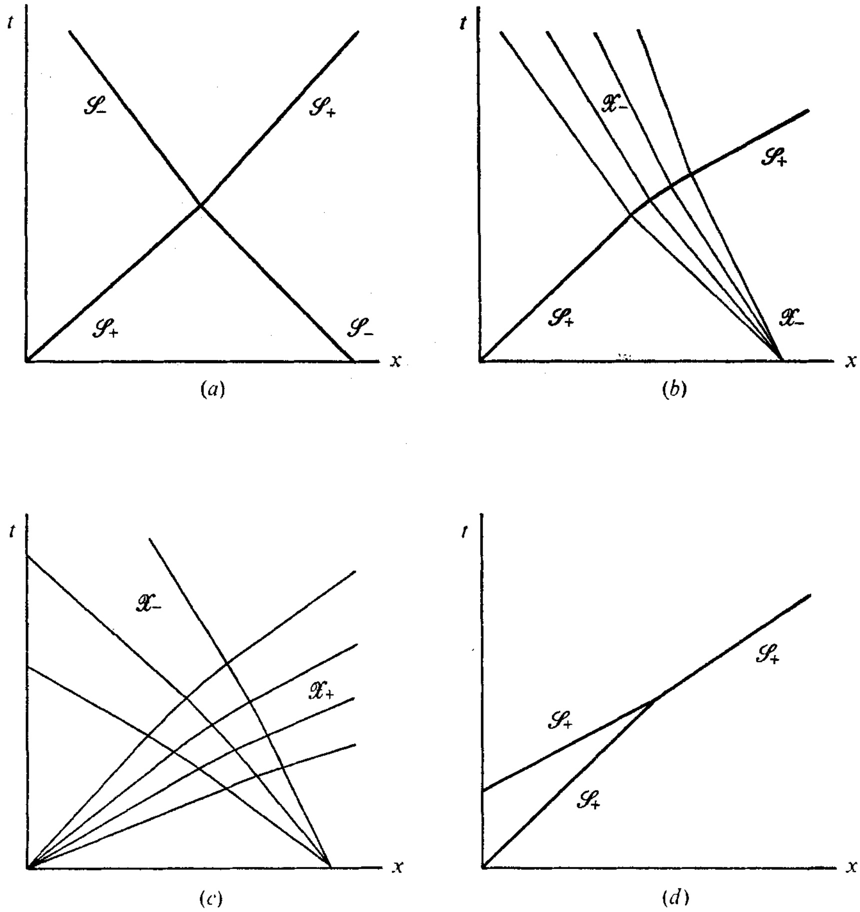

FIGURE 2. Interaction of weak shocks and expansion waves: ▬▬, shocks; ——, expansion waves.

duces either a single shock or an expansion wave, depending on the relative strengths of the original waves. The strength of the resulting wave is found to be equal to the difference of the strengths of the original waves.

All the results described above concerning the interactions of weak nonlinear waves in a compressible fluid have been derived by rather heuristic arguments, (i) assuming that the disturbance fields are composed of shocks, expansion waves and contact surfaces, connected by regions of constant state, and (ii) neglecting the inner structure of shocks and contact surfaces. These results, however, can be obtained more systematically and rigorously by solving the fundamental equations of a compressible fluid under proper approximations corresponding to weak nonlinear disturbances. It will be shown in § 3 that a reductive perturbation method applied to the equations of conservation of mass, momentum and energy and the equation of state of a compressible fluid leads to more definite results than those described in this section.

3. Nonlinear waves in a compressible, viscous and heat-conducting fluid

One-dimensional motions in a compressible fluid with finite viscosity and thermal conductivity are governed by the following equations, representing the conservation of mass, momentum and energy:

$$\left.\begin{aligned} \frac{D\rho}{Dt}+\rho\frac{\partial u}{\partial x} &= 0,\\ \rho\frac{Du}{Dt}+\frac{\partial}{\partial x}\left\{p-(\tfrac{4}{3}\eta+\zeta)\frac{\partial u}{\partial x}\right\} &= 0,\\ \rho\frac{DE}{Dt}+\left\{p-(\tfrac{4}{3}\eta+\zeta)\frac{\partial u}{\partial x}\right\}\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) &= 0, \end{aligned}\right\} \tag{3.1}$$

with

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x},$$

where η and ζ denote the shear and bulk viscosities respectively and k the thermal conductivity.

To make the above system of equations more tractable, we introduce some simplifying assumptions (following Lighthill 1956). First, all the transport coefficients are assumed to be constant in space; and their ratios to the density, η/ρ, ζ/ρ and k/ρ, are replaced by the corresponding values η_0/ρ_0, ζ_0/ρ_0 and k_0/ρ_0 in the undisturbed state. Next, the viscous dissipation $(\frac{4}{3}\eta+\zeta)(\partial u/\partial x)^2$ in (3.1) is neglected. Concerning the nature and the validity of these approximations, reference may be made to Lighthill (1956).

Then, eliminating p and E from (3.1), (2.2) and (2.3), and applying the above approximations, we have the following equations for ρ, u and T:

$$\left.\begin{aligned} \frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x} &= 0,\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathscr{R}\frac{T}{\rho}\frac{\partial\rho}{\partial x}+\mathscr{R}\frac{\partial T}{\partial x}-\delta_1\frac{\partial^2 u}{\partial x^2} &= 0,\\ \frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+(\gamma-1)T\frac{\partial u}{\partial x}-\delta_2\frac{\partial^2 T}{\partial x^2} &= 0, \end{aligned}\right\} \tag{3.2}$$

where

$$\delta_1=\frac{1}{\rho_0}(\tfrac{4}{3}\eta_0+\zeta_0),\quad \delta_2=\frac{k_0}{c_V\rho_0}. \tag{3.3}$$

For later convenience, all the variables are made non-dimensional, using a representative length scale L, density ρ_0, temperature T_0 and sound velocity $a_0=(\gamma p_0/\rho_0)^{\frac{1}{2}}=(\gamma\mathscr{R}T_0)^{\frac{1}{2}}$ in the undisturbed fluid:

$$x'=\frac{x}{L},\quad t'=\frac{t}{L/a_0},\quad u'=\frac{u}{a_0},\quad \rho'=\frac{\rho}{\rho_0},\quad T'=\frac{T}{T_0}. \tag{3.4}$$

Then (3.2) may be written in non-dimensional form as

$$\frac{\partial W}{\partial t'}+A(W)\frac{\partial W}{\partial x'}-K\frac{\partial^2 W}{\partial x'^2}=0, \tag{3.5}$$

where

$$\left.\begin{aligned} W &= \begin{pmatrix} \rho' \\ u' \\ T' \end{pmatrix}, \quad A = \begin{pmatrix} u' & \rho' & 0 \\ \dfrac{1}{\gamma}\dfrac{T'}{\rho'} & u' & \dfrac{1}{\gamma} \\ 0 & (\gamma-1)T' & u' \end{pmatrix}, \\ K &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{1}{R_1} & 0 \\ 0 & 0 & \dfrac{1}{R_2} \end{pmatrix}, \end{aligned}\right\} \tag{3.6}$$

and $R_1 = a_0 L/\delta_1$ and $R_2 = a_0 L/\delta_2$ are non-dimensional numbers similar to the usual Reynolds number.

A singular perturbation method for solving a wide class of nonlinear partial differential equations was presented by Taniuti & Wei (1968). This method enables us to express each quasi-simple wave solution of the equations, which propagates along a family of characteristics, as the solution of a single nonlinear equation reduced from the original equations. Using this method, Oikawa & Yajima (1973) dealt with the interaction of solitary waves belonging to different families of characteristics, and calculated the phase shifts of the two solitary waves due to collision in the cases of ion-acoustic waves in a collisionless plasma and shallow-water waves. The same method can be applied to the present problem, in which, however, the interaction of nonlinear waves will cause changes in the phase velocities, rather than mere phase shifts. In the following, we shall investigate the interaction of weak nonlinear waves in a compressible fluid governed by (3.5), using this reductive perturbation method.

In the unperturbed state, W and A of (3.6) are written as

$$W^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad A^{(0)} = A(W^{(0)}) = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{1}{\gamma} & 0 & \dfrac{1}{\gamma} \\ 0 & \gamma-1 & 0 \end{pmatrix}. \tag{3.7}$$

The eigenvalues r of $A^{(0)}$ defined by the roots of

$$|A^{(0)} - rI| = 0,$$

I being the unit matrix, are found to be all real and distinct:

$$r_1 = 1, \quad r_2 = -1, \quad r_3 = 0. \tag{3.8}$$

Now we expand W and A in powers of a smallness parameter ϵ, which represents the order of magnitude of the amplitude of nonlinear waves:

$$\left.\begin{aligned} W &= W^{(0)} + \epsilon W^{(1)} + \epsilon^2 W^{(2)} + \dots, \\ A &= A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \dots, \end{aligned}\right\} \tag{3.9}$$

and regard all W's and A's as functions of the new variables

$$\left.\begin{aligned} \xi_j &= \epsilon[x' - r_j t' - \phi_j(x', t')], \quad j = 1, 2, 3, \\ \tau &= \epsilon^2 t', \end{aligned}\right\} \tag{3.10}$$

where the ϕ_j are phase functions to be determined later. The transformation (3.10) has been suggested from the functional form of the well-known weak-shock solution (4.2), and also from the fact that weak shocks change their phase velocities after a head-on collision. It may be said that, with these specific expansions and stretching of variables, a consistent set of equations are obtained for examining the large-scale structure of nonlinear waves and their behaviour after a long time. Substituting the expansion (3.9) with (3.10) into (3.5), and equating terms of the same powers of ϵ to zero, we obtain a sequence of equations.

At the lowest order we have

$$\sum_j (A^{(0)} - r_j I) \frac{\partial W^{(1)}}{\partial \xi_j} = 0. \tag{3.11}$$

Denote the right and left eigenvectors of $A^{(0)}$ for the eigenvalue r_i by Q_i and P_i, respectively:

$$\left.\begin{aligned} A^{(0)} Q_i &= r_i Q_i, \quad P_i A^{(0)} = r_i P_i, \\ (P_i, Q_j) &= \delta_{ij}, \quad i, j = 1, 2, 3. \end{aligned}\right\} \tag{3.12}$$

Then, expanding $W^{(1)}$ in terms of Q_i as

$$W^{(1)} = \sum_i f_i(\xi_1, \xi_2, \xi_3; \tau) Q_i, \tag{3.13}$$

and substituting (3.13) into (3.11), we obtain

$$\sum_j (r_i - r_j) \frac{\partial f_i}{\partial \xi_j} = 0. \tag{3.14}$$

The general solution of (3.14) is expressed as a function of

$$\xi_{ij} = \xi_j - (r_i - r_j) \sum_k (r_i - r_k) \xi_k / \sum_k (r_i - r_k)^2 \tag{3.15}$$

and τ, where ξ_{ij} is the component vector of ξ_j perpendicular to the vector $(r_i - r_j)$. Since, however, we are interested in quasi-simple waves which propagate along a family of characteristics ξ_j = const., we take f_i as a function of $\xi_{ii} = \xi_i$ alone:

$$f_i = f_i(\xi_i, \tau). \tag{3.16}$$

At the next order we have

$$\sum_j (A^{(0)} - r_j I) \frac{\partial W^{(2)}}{\partial \xi_j} - \sum_j \sum_k (A^{(0)} - r_k I) \frac{\partial \phi_j}{\partial \xi_k} \frac{\partial W^{(1)}}{\partial \xi_j} + \frac{\partial W^{(1)}}{\partial \tau} + A^{(1)} \sum_j \frac{\partial W^{(1)}}{\partial \xi_j} - K \sum_j \frac{\partial^2 W^{(1)}}{\partial \xi_j^2} = 0, \tag{3.17}$$

where $A^{(1)}$ may be expressed as

$$A^{(1)} = [(W^{(1)} . \nabla_W) A]^{(0)}, \quad \nabla_W = (\partial/\partial\rho, \partial/\partial u, \partial/\partial T).$$

Then, expanding $W^{(2)}$ and $A^{(1)}$ in terms of Q_i as

$$\left.\begin{aligned} W^{(2)} &= \sum_i g_i(\xi_1, \xi_2, \xi_3; \tau)\, Q_i, \\ A^{(1)} &= \sum_i f_i(\xi_i, \tau)\, [(Q_i . \nabla_W) A]^{(0)}, \end{aligned}\right\} \tag{3.18}$$

and substituting (3.13) and (3.18) into (3.17), we obtain

$$\sum_{j \neq i} \left[(r_i - r_j) \frac{\partial g_i}{\partial \xi_j} + M_{ikj} f_k \frac{\partial f_j}{\partial \xi_j} - K_{ij} \frac{\partial^2 f_j}{\partial \xi_j^2}\right] + \sum_{j \neq i} \left[(r_j - r_i) \frac{\partial \phi_i}{\partial \xi_j} + M_{iji} f_j\right] \frac{\partial f_i}{\partial \xi_i} + G_i[f_i] = 0, \tag{3.19}$$

with

$$\left.\begin{aligned} M_{ijk} &= (P_i, [(Q_j . \nabla_W) A]^{(0)} Q_k), \\ K_{ij} &= (P_i, K Q_j), \\ G_i[f_i] &= \frac{\partial f_i}{\partial \tau} + M_{iii} f_i \frac{\partial f_i}{\partial \xi_i} - K_{ii} \frac{\partial^2 f_i}{\partial \xi_i^2}, \end{aligned}\right\} \tag{3.20}$$

where (3.16) has been taken into account.

Now we require that the phase functions ϕ_i satisfy the equations

$$\sum_{j \neq i} \left[(r_j - r_i) \frac{\partial \phi_i}{\partial \xi_j} + M_{iji} f_j\right] = 0,$$

the solutions of which are given immediately by

$$\phi_i = \sum_{j \neq i} \frac{M_{iji}}{r_i - r_j} \int^{\xi_j} f_j(\xi, \tau)\, d\xi + \theta_i(\xi_{i1}, \xi_{i2}, \xi_{i3}; \tau), \tag{3.21}$$

where the θ_i are arbitrary functions to be determined by the initial conditions. With the phase functions ϕ_i thus determined, the second sum of (3.19) vanishes, and then it may easily be seen that integration of (3.19) produces secular terms for g_i, unless the terms $G_i[f_i]$, being constant in integration with respect to ξ_j, $j \neq i$, vanish identically. Thus it follows from (3.20) that

$$\frac{\partial f_i}{\partial \tau} + M_{iii} f_i \frac{\partial f_i}{\partial \xi_i} - K_{ii} \frac{\partial^2 f_i}{\partial \xi_i^2} = 0, \tag{3.22}$$

which give the reduced equations for $f_i(\xi_i, \tau)$.

In the present problem, the eigenvectors are given, from (3.7), (3.8) and (3.12), as

$$Q_1 = \begin{pmatrix} 1 \\ 1 \\ \gamma - 1 \end{pmatrix}, \quad P_1 = \left(\frac{1}{2\gamma}, \frac{1}{2}, \frac{1}{2\gamma}\right), \tag{3.23a}$$

$$Q_2 = \begin{pmatrix} 1 \\ -1 \\ \gamma - 1 \end{pmatrix}, \quad P_2 = \left(\frac{1}{2\gamma}, -\frac{1}{2}, \frac{1}{2\gamma}\right), \tag{3.23b}$$

$$Q_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad P_3 = \left(\frac{\gamma - 1}{\gamma}, 0, -\frac{1}{\gamma}\right), \tag{3.23c}$$

and hence

$$\left.\begin{aligned}
&M_{111} = -M_{222} = \tfrac{1}{2}(\gamma+1), \quad M_{333} = 0,\\
&M_{121} = -M_{212} = \tfrac{1}{2}(\gamma-3), \quad M_{131} = -M_{232} = -\tfrac{1}{2},\\
&M_{313} = -M_{323} = 1,\\
&K_{11} = K_{22} = \tfrac{1}{2}\left(\frac{1}{R_1}+\frac{\gamma-1}{\gamma}\frac{1}{R_2}\right) = \frac{1}{2\beta R},\\
&K_{33} = \frac{1}{\gamma R_2} = \frac{1}{\sigma R},
\end{aligned}\right\} \tag{3.23d}$$

where

$$\left.\begin{aligned}
&\beta = \tfrac{4}{3}+\frac{\zeta_0}{\eta_0}+\frac{\gamma-1}{\sigma},\\
&\sigma = \frac{\eta_0 c_p}{k_0} \quad \text{(the Prandtl number)},\\
&R = \frac{a_0 L}{\eta_0/\rho_0} \text{ (the Reynolds number)}.
\end{aligned}\right\} \tag{3.24}$$

Then, the reduced equations (3.22) are written as

$$\left.\begin{aligned}
&\frac{\partial F_i}{\partial \tau}+F_i\frac{\partial F_i}{\partial \xi_i}-\frac{1}{2\beta R}\frac{\partial^2 F_i}{\partial \xi_i^2} = 0, \quad i = 1 \text{ and } 2,\\
&f_i = (-1)^{i-1}\frac{2}{\gamma+1}F_i(\xi_i,\tau),
\end{aligned}\right\} \tag{3.25}$$

$$\frac{\partial f_3}{\partial \tau}-\frac{1}{\sigma R}\frac{\partial^2 f_3}{\partial \xi_3^2} = 0, \tag{3.26}$$

the first of which is called the Burgers equation, the last being the equation of heat conduction. The phase functions (3.21) are also written as

$$\left.\begin{aligned}
&\phi_1 = -\frac{1}{2}\frac{\gamma-3}{\gamma+1}\int^{\xi_2}F_2(\xi)\,d\xi-\frac{1}{2}\int^{\xi_3}f_3(\xi)\,d\xi+\theta_1,\\
&\phi_2 = \frac{1}{2}\frac{\gamma-3}{\gamma+1}\int^{\xi_1}F_1(\xi)\,d\xi-\frac{1}{2}\int^{\xi_3}f_3(\xi)\,d\xi+\theta_2,\\
&\phi_3 = -\frac{2}{\gamma+1}\int^{\xi_1}F_1(\xi)\,d\xi+\frac{2}{\gamma+1}\int^{\xi_2}F_2(\xi)\,d\xi+\theta_3.
\end{aligned}\right\} \tag{3.27}$$

Lastly, the first-order solution $W^{(1)}$ of (3.5) is obtained:

$$W^{(1)} = \begin{pmatrix}\rho^{(1)'}\\ u^{(1)'}\\ T^{(1)'}\end{pmatrix} = \frac{2}{\gamma+1}F_1(\xi_1,\tau)\begin{pmatrix}1\\ 1\\ \gamma-1\end{pmatrix}-\frac{2}{\gamma+1}F_2(\xi_2,\tau)\begin{pmatrix}1\\ -1\\ \gamma-1\end{pmatrix}+f_3(\xi_3,\tau)\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}. \tag{3.28}$$

Thus, the nonlinear waves in a compressible fluid are expressed as linear combinations of three quasi-simple waves, which are governed by the reduced equations (3.25) and (3.26), and the interaction between them occurs only through the phase functions (3.27).

4. Interaction of shocks and expansion waves

Let us assume that, at an initial instant, there are only shocks and expansion waves, and no contact surface, present. Then, since the solution f_3 of (3.26), which was initially absent, must identically vanish, the expression (3.28) for the first-order disturbance is reduced to

$$W^{(1)} = \begin{pmatrix} \rho^{(1)'} \\ u^{(1)'} \end{pmatrix} = \frac{2}{\gamma+1}\left[F_1(\xi_1,\tau)\begin{pmatrix}1\\1\end{pmatrix} - F_2(\xi_2,\tau)\begin{pmatrix}1\\-1\end{pmatrix}\right]. \tag{4.1}$$

4.1. *Head-on collision of two shocks*

The solution of (3.25), which represents a shock wave, was obtained by Burgers (1950):

$$F_i = V_i - \tfrac{1}{2}v_i \tanh(\tfrac{1}{2}\beta R v_i X_i), \quad X_i = \xi_i - V_i\tau, \tag{4.2}$$

where V_i ($\gtrless 0$) and v_i (>0) are arbitrary constants. The expression (4.2) gives a negative step function connecting a constant region $F_i = V_i + \frac{1}{2}v_i$ to the left and another constant region $F_i = V_i - \frac{1}{2}v_i$ to the right of the step, and moving with a constant velocity V_i along the ξ_i axis. Indeed, this solution (4.2) is nothing but the weak-shock solution first obtained by Taylor (1910).

The phase functions corresponding to the solution (4.2) are obtained from (3.27):

$$\left.\begin{aligned} \phi_1 &= \theta_1 - \frac{1}{2}\frac{\gamma-3}{\gamma+1}\left[V_2X_2 - \frac{1}{\beta R}\log\cosh(\tfrac{1}{2}\beta R v_2 X_2)\right], \\ \phi_2 &= \theta_2 + \frac{1}{2}\frac{\gamma-3}{\gamma+1}\left[V_1X_1 - \frac{1}{\beta R}\log\cosh(\tfrac{1}{2}\beta R v_1 X_1)\right]. \end{aligned}\right\} \tag{4.3}$$

For extremely large Reynolds numbers R, (4.3) reduces to

$$\left.\begin{aligned} \phi_1 &= \theta_1 - \frac{1}{2}\frac{\gamma-3}{\gamma+1}\begin{cases}(V_2+\frac{1}{2}v_2)X_2 & \text{for } X_2<0,\\ (V_2-\frac{1}{2}v_2)X_2 & \text{for } X_2>0,\end{cases} \\ \phi_2 &= \theta_2 + \frac{1}{2}\frac{\gamma-3}{\gamma+1}\begin{cases}(V_1+\frac{1}{2}v_1)X_1 & \text{for } X_1<0,\\ (V_1-\frac{1}{2}v_1)X_1 & \text{for } X_1>0.\end{cases} \end{aligned}\right\} \tag{4.4}$$

For definiteness, let us take the case dealt with in §2 (see figure 2(*a*)), in which the fluid ahead of both shocks is in undisturbed state, so that

$$v_1 = 2V_1 = \frac{\gamma+1}{2\epsilon}u_1', \quad -v_2 = 2V_2 = \frac{\gamma+1}{2\epsilon}u_2', \tag{4.5}$$

where (3.9), (3.13), (3.25) and (4.2) have been taken into account. The paths of shocks $\mathscr{S}_+$ and $\mathscr{S}_-$ are given by $X_1 = 0$ and $X_2 = 0$, respectively; and in this case X_1 and X_2 are expressed from (4.2) and (4.4) as follows. Before collision,

$$\left.\begin{aligned} \phi_1 &= 0, \quad X_1 = \epsilon[x-(1+\epsilon V_1)t] \quad \text{for } \mathscr{S}_+(X_2<0),\\ \phi_2 &= 0, \quad X_2 = \epsilon[x-(-1+\epsilon V_2)t] \quad \text{for } \mathscr{S}_-(X_1>0).\end{aligned}\right\} \tag{4.6}$$

After collision,

$$\left.\begin{aligned}
\phi_1 &= \frac{1}{2}\frac{\gamma-3}{\gamma+1}v_2\epsilon[x-(-1+\epsilon V_2)t-\phi_2] \\
&= \frac{1}{2}\frac{\gamma-3}{\gamma+1}\epsilon v_2(x+t)+O(\epsilon^2), \\
X_1 &= \epsilon[x-(1+\epsilon V_1)t-\phi_1] \\
&= \epsilon\left(1-\frac{1}{2}\frac{\gamma-3}{\gamma+1}\epsilon v_2\right)\left[x-\left(1+\epsilon V_1+\frac{\gamma-3}{\gamma+1}\epsilon v_2\right)t\right] \\
&\quad +O(\epsilon^3) \quad \text{for} \quad \mathscr{S}_+\ (X_2>0); \\
\phi_2 &= \frac{1}{2}\frac{\gamma-3}{\gamma+1}v_1\epsilon[x-(1+\epsilon V_1)t-\phi_1] \\
&= \frac{1}{2}\frac{\gamma-3}{\gamma+1}\epsilon v_1(x-t)+O(\epsilon^2), \\
X_2 &= \epsilon[x-(-1+\epsilon V_2)t-\phi_2] \\
&= \epsilon\left(1-\frac{1}{2}\frac{\gamma-3}{\gamma+1}\epsilon v_1\right)\left[x-\left(-1+\epsilon V_2-\frac{\gamma-3}{\gamma+1}\epsilon v_1\right)t\right] \\
&\quad +O(\epsilon^3) \quad \text{for} \quad \mathscr{S}_-\ (X_1<0).
\end{aligned}\right\} \tag{4.7}$$

In view of (4.5), the phase velocities of $\mathscr{S}_+$ and $\mathscr{S}_-$ before and after collision are in complete agreement with those given by (2.16) and (2.17). It may also be seen, from (4.6) and (4.7), that the collision has the effect of increasing the Reynolds number of shocks $\mathscr{S}_+$ and $\mathscr{S}_-$ by the factors $\{1+[\frac{1}{2}(3-\gamma)(\gamma+1)]\epsilon v_2\}$ and $\{1+[\frac{1}{2}(3-\gamma)(\gamma+1)]\epsilon v_1\}$, respectively.

For illustrating the situation, the result of numerical calculation of the head-on collision of two shocks is shown graphically in figure 3(*a*).

4.2. *Head-on collision of a shock and an expansion wave*

Let us consider a forward-facing shock $\mathscr{S}_+$, given by F_1 of (4.2), and a backward-facing expansion wave $\mathscr{X}_-$, expressed by

$$F_2 = \begin{cases} V_2-\frac{1}{2}v_2 & \text{for} \quad X_2 < -\frac{1}{2}v_2\tau, \\ V_2+X_2/\tau & \text{for} \quad -\frac{1}{2}v_2\tau < X_2 < \frac{1}{2}v_2\tau, \\ V_2+\frac{1}{2}v_2 & \text{for} \quad X_2 < \frac{1}{2}v_2\tau. \end{cases} \tag{4.8}$$

The expansion wave (4.8) occupies a length $v_2\tau$ of the ξ_2 axis, which increases with time, and connects a constant region $V_2-\frac{1}{2}v_2$ to the left and another constant region $V_2+\frac{1}{2}v_2$ to the right.

Obviously, the phase function ϕ_2 is identical with that given by (4.3) or (4.4), whereas ϕ_1 is derived from (3.27):

$$\phi_1 = \theta_1-\frac{1}{2}\frac{\gamma-3}{\gamma+1}\begin{cases} (V_2-\frac{1}{2}v_2)X_2-\frac{1}{8}v_2^2 & \text{for} \quad X_2<-\frac{1}{2}v_2\tau, \\ V_2X_2+X_2^2/\tau & \text{for} \quad -\frac{1}{2}v_2\tau \leqslant X_2 < \frac{1}{2}v_2\tau, \\ (V_2+\frac{1}{2}v_2)X_2-\frac{1}{8}v_2^2 & \text{for} \quad X_2 \geqslant \frac{1}{2}v_2\tau. \end{cases} \tag{4.9}$$

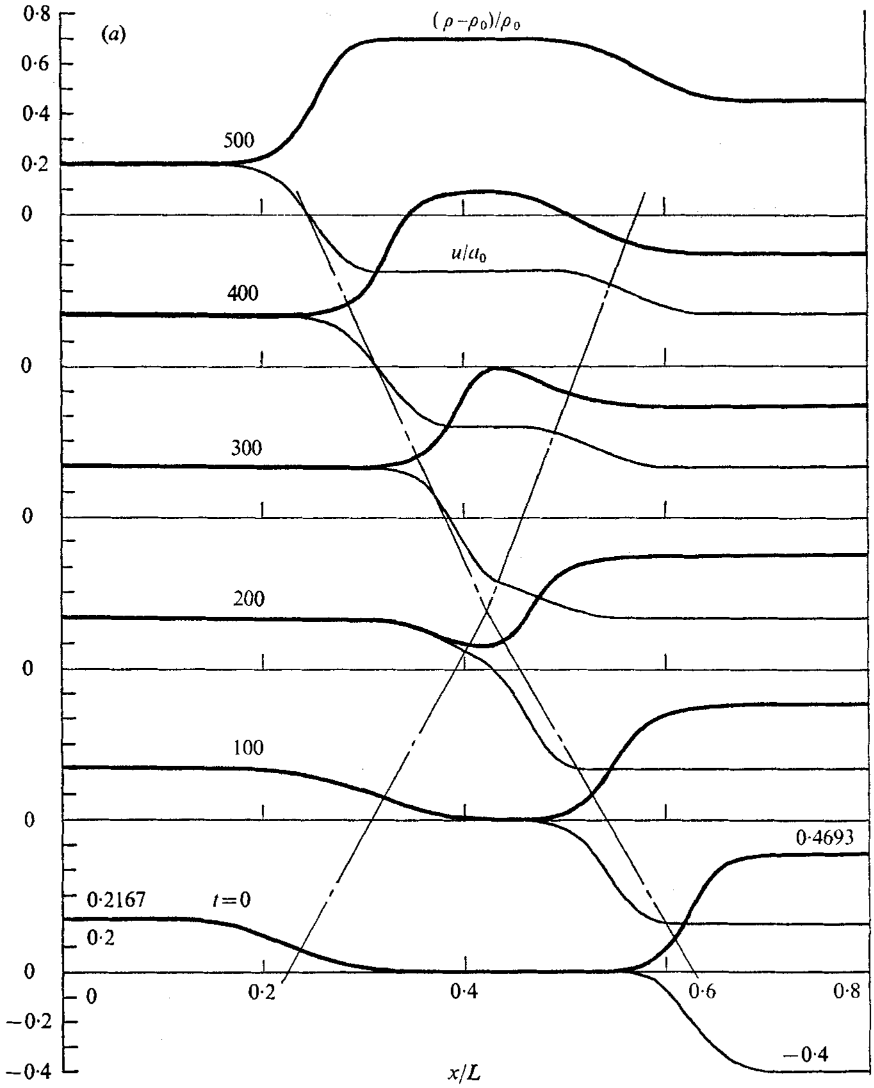

FIGURE 3(*a*). For legend see next page.

In the special case of $V_2 = \frac{1}{2}v_2$, (4.9) becomes

$$\phi_1 = -\frac{1}{2}\frac{\gamma-3}{\gamma+1}\begin{cases} 0 & \text{for} \quad X_2 < -\frac{1}{2}v_2\tau, \\ \frac{1}{2}v_2X_2 + X_2^2/\tau + \frac{1}{8}v_2^2 & \text{for} \quad -\frac{1}{2}v_2\tau \leqslant X_2 < \frac{1}{2}v_2\tau, \\ v_2X_2 & \text{for} \quad X_2 \geqslant \frac{1}{2}v_2\tau, \end{cases} \tag{4.10}$$

where θ_1 has been chosen so as to make ϕ_1 vanish at $X_2 < -\frac{1}{2}v_2\tau$.

Thus the shock $\mathscr{S}_+$ is continuously accelerated by passing through the region of the expansion wave $\mathscr{X}_-$, and eventually acquires an increment

$$[\tfrac{1}{2}(3-\gamma)/(\gamma+1)]\,\epsilon v_2$$

in the phase velocity. The expansion wave $\mathscr{X}_-$, on the other hand, is decelerated

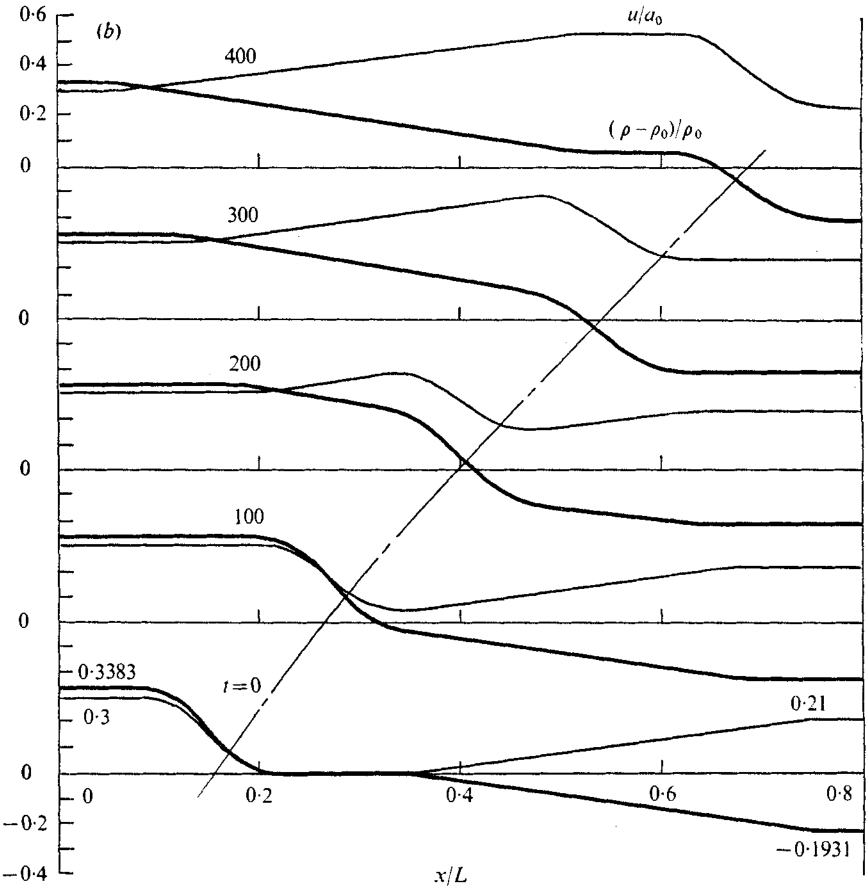

FIGURE 3. (*a*) Head-on collision of two weak shocks. (*b*) Head-on collision of a weak shock and an expansion wave. ▬▬, density profiles $(\rho-\rho_0)/\rho_0$; ——, velocity profiles u/a_0; — – —, loci of shock fronts.

by the collision in the same manner as the shock $\mathscr{S}_-$ in § 4.1. The situation is illustrated in figure 3(*b*), which shows the result of numerical calculation of the head-on collision of a shock and an expansion wave.

4.3. *Overtaking of like-facing waves*

In the case of interaction of shocks and expansion waves facing in the same direction the disturbance field is completely determined by (3.25); therefore the problem is reduced to the initial-value problem of the Burgers equation. It will be shown in § 5 that the general solution of the Burgers equation started from an arbitrary initial condition tends, for very large Reynolds numbers $R \gg 1$ and times $\tau \gg 1$, to a sequence of triangular shock waves, each consisting of a shock front followed by a simple expansion wave.†

† The formation of a series of triangular shock waves and the coalescence of successive shock fronts on overtaking were first pointed out by Burgers (1950) for the solution of the Burgers equation starting from a broken linear velocity profile and in the limit of infinite Reynolds number; later the proof was extended to the case of an arbitrary initial condition by Burgers (1954*a*–*c*, 1972, 1973).

The overtaking of one step shock by another can be dealt with as a limiting case of the general result, which will be given in § 5; the corresponding solution is found to be expressed as follows. Before overtaking ($\tau < 0$),

$$F_i(\xi_i, \tau) = \begin{cases} V_{i1} - \tfrac{1}{2}v_{i1} \tanh[\tfrac{1}{2}\beta R v_{i1}(\xi_i - V_{i1}\tau)] & \text{for} \quad \xi_i < 0, \\ V_{i2} - \tfrac{1}{2}v_{i2} \tanh[\tfrac{1}{2}\beta R v_{i2}(\xi_i - V_{i2}\tau)] & \text{for} \quad \xi_i > 0, \end{cases} \tag{4.11}$$

where $V_{i1} - \frac{1}{2}v_{i1} = V_{i2} + \frac{1}{2}v_{i2}$. After overtaking ($\tau > 0$),

$$F_i(\xi_i, \tau) = (V_{i1} + V_{i2}) - \tfrac{1}{2}(v_{i1} + v_{i2}) \tanh\{\tfrac{1}{2}\beta R(v_{i1} + v_{i2})[\xi_i - (V_{i1} + V_{i2})\tau]\}. \tag{4.12}$$

Thus, unlike the head-on collision, the overtaking of one shock by another makes the two coalesce, to form a single shock with strength $(v_{i1} + v_{i2})$ and phase-velocity $(V_{i1} + V_{i2})$.

The overtaking that takes place between a shock and an expansion wave can be dealt with in a similar manner. The general feature of the interaction may be said to be that the shock and expansion wave are diminished through the process of overtaking; and eventually there remains only a shock or an expansion wave, whose strength is given by the difference of the strengths of the original waves.

5. One-dimensional shock turbulence

The results established in §§ 2–4 may be summarized as follows. (i) If the disturbance field in a compressible fluid initially involves only weak shocks and expansion waves, it remains composed only of shocks and expansion waves at all times. (ii) The disturbance field is divided into two families, of forward- and backward-facing waves. Each family, consisting of shocks and expansion waves, is independently governed by the Burgers equation; and the influence of the other family appears only through the phase function in the argument. (iii) Consequently, the asymptotic form of a disturbance field of arbitrary initial form at very large Reynolds numbers and times is expressed as the sum of the asymptotic forms of solutions of the Burgers equations corresponding to the forward- and backward-facing waves.

Thus, if we are interested in the asymptotic behaviour of a random disturbance in a compressible fluid, or compression turbulence, at very large Reynolds numbers and times, we need only investigate the corresponding behaviour of the solutions of the Burgers equation. The general solution of the Burgers equation (3.25) was given by Hopf (1950) and Cole (1951) as

$$F_i(\xi_i, \tau) = -\frac{1}{\beta R}\frac{\partial}{\partial \xi_i}\log \Theta(\xi_i, \tau), \tag{5.1}$$

where Θ is the general solution of the heat-conduction equation

$$\frac{\partial \Theta}{\partial \tau} - \frac{1}{2\beta R}\frac{\partial^2 \Theta}{\partial \xi_i^2} = 0. \tag{5.2}$$

It was shown by Tatsumi & Kida (1972), and independently by Burgers (1972, 1973), that, for very large values of R and τ, the solution of the Burgers equation

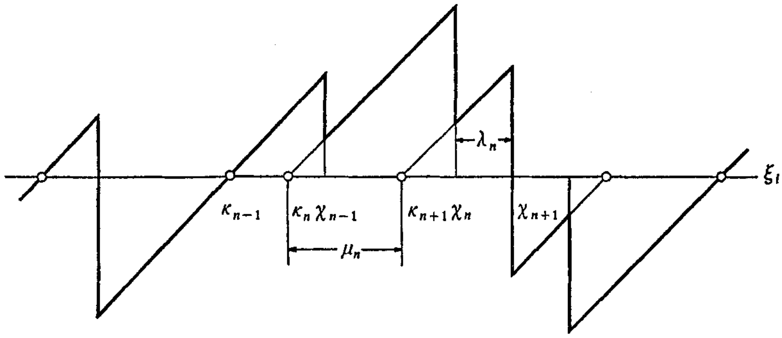

FIGURE 4. Sequence of triangular shock waves.

(3.25), started from an arbitrary initial condition, tends asymptotically to a sequence of triangular shock waves:†

$$F_i(\xi_i,\tau) = \frac{1}{\tau}[\xi_i - \tfrac{1}{2}(\kappa_n+\kappa_{n+1})] - \frac{1}{2\tau}(\kappa_{n+1}-\kappa_n)\tanh\left[\frac{\beta R}{2\tau}(\kappa_{n+1}-\kappa_n)(\xi_i-\chi_n)\right], \tag{5.3}$$

$$\frac{d\chi_n}{d\tau} = \frac{1}{\tau}[\chi_n - \tfrac{1}{2}(\kappa_n+\kappa_{n+1})], \tag{5.4}$$

at $\frac{1}{2}(\chi_{n-1}+\chi_n) < \xi_i < \frac{1}{2}(\chi_n+\chi_{n+1})$, $n = 1, 2, \ldots, N$. In (5.3) and (5.4), the suffix i has been omitted for χ's and κ's; χ_n denotes the moving co-ordinate of the nth shock front and κ_n the stationary co-ordinate of the intersection of the straight line representing the expansion wave with the ξ_i axis as depicted in figure 4. Each triangular shock of (5.3) has strength

$$v_{in} = \frac{1}{\tau}(\kappa_{n+1}-\kappa_n), \tag{5.5}$$

which decreases in time, and phase velocity

$$V_{in} = \frac{1}{\tau}[\chi_n - \tfrac{1}{2}(\kappa_n+\kappa_{n+1})], \tag{5.6}$$

which is readily shown to be constant in time.

Since the velocities V_{in} of the shock fronts are in general not the same, overtaking of one shock by another takes place continuously and everywhere, unless the number density of shock fronts is too small. If we examine the asymptotic behaviour of the solution of the Burgers equation, which eventually leads to (5.3), we find that the single shock that emerges from the overtaking of the $(n+1)$th shock by the nth shock is expressed by (5.3) and (5.4), with the suffix $n+1$ replaced by $n+2$. Hence, in view of (5.5) and (5.6), the strength of the new shock is given by the sum of the strengths of the original shocks,

$$\frac{1}{\tau}(\kappa_{n+2}-\kappa_n) = v_{in} + v_{i(n+1)}; \tag{5.7}$$

† For earlier works of Burgers related to this problem, see the footnote in §4.3. There is another group of papers, dealing with Burgers model of turbulence for a channel flow by solving a set of model equations for the mean flow and the disturbances (Case & Chiu 1969; Lee 1971; Murray 1973). Some of these also discuss the formation of triangular waves, and their coalescence in a channel.

and its phase velocity is given by the weighted mean of the original velocities,

$$\frac{1}{\tau}[\chi_n - \tfrac{1}{2}(\kappa_n + \kappa_{n+2})] = \frac{v_{in} V_{in} + v_{i(n+1)} V_{i(n+1)}}{v_{in} + V_{i(n+1)}}. \tag{5.8}$$

Burgers (1950) suggested that these relations of coalescence are more clearly understood by regarding each shock front as a particle of the mass v_{in} moving with the velocity V_{in}. Then the relations (5.7) and (5.8) are nothing but the well-known laws of conservation of the 'mass' v_{in} and the 'momentum' $v_{in} V_{in}$ in a collision, which is in this case perfectly 'inelastic'.

Incidentally, the expression of F_i for a step shock is easily derived from that for a triangular shock (5.3) and (5.4) by taking the limit $\tau \to \infty$, $\kappa_{n+1} - \kappa_n \to \infty$, but keeping $\xi_i - \chi_n$, v_{in} and V_{in} finite. In particular, the first expression of (4.11) is obtained as the limit of (5.3) for $n = 1$, the second by that for $n = 2$, and (4.12) by that for $n = 1$, with $n+1$ replaced by $n+2$.

According to the properties of weak nonlinear waves (i)–(iii), summarized at the beginning of this section, the forward- and backward-facing components of weak-shock turbulence behave independently of each other; and they are governed by the Burgers equation (3.25) so long as they are expressed in terms of the ξ_i co-ordinates. Hence, their statistical properties also, when described in ξ_i, τ space, are identical with those of the Burgers model of turbulence.

The statistical mechanics of Burgers turbulence, represented by a train of triangular shock waves or a 'gas of adhesive particles', was investigated by Tatsumi & Kida (1972). The equations that govern the distribution functions of the strengths $\mu_n = \kappa_{n+1} - \kappa_n$ and the intervals $\lambda_n = \chi_{n+1} - \chi_n$ of shock fronts, were derived, and the self-preserving solutions were obtained. As a consequence, the turbulent energy was shown to decay in time as $t^{2(\alpha-1)}$ $(0 \leqslant \alpha < 1)$, involving two important special cases $t^{-\frac{2}{3}}$ and t^{-1}, the latter of which is in fairly good agreement with the result of the numerical experiment of Crow & Canavan (1970).

When we come to expressing the final results in the physical (x, t) space, however, we have to deal with the phase functions ϕ_i, which are included in the transformation $\xi_i = \epsilon(x - r_i t - \phi_i)$. Nevertheless, it will be shown below that ϕ_i can be neglected even in x, t space, as far as the quantities expressed as space averages are concerned.

When the forward- and backward-facing components of turbulence take the form of a train of triangular shocks (5.3), the phase functions (3.27) are written as

$$\left.\begin{aligned} \phi_1 &= \phi_1(\tfrac{1}{2}(\chi_{n-1} + \chi_n)) - \tfrac{1}{2}\frac{\gamma-3}{\gamma+1}\left\{\frac{1}{2\tau}[\xi - \tfrac{1}{2}(\kappa_n + \kappa_{n+1})]^2 \right. \\ &\qquad \left. - \frac{1}{\beta R}\log\cosh\left[\frac{\beta R}{2\tau}(\kappa_{n+1} - \kappa_n)(\xi - \chi_n)\right]\right\}_{\xi = \frac{1}{2}(\chi_{n-1}+\chi_n)}^{\xi = \xi_2}, \\ \phi_2 &= \phi_2(\tfrac{1}{2}(\chi_{n-1} + \chi_n)) + \tfrac{1}{2}\frac{\gamma-3}{\gamma+1}\left\{\frac{1}{2\tau}[\xi - \tfrac{1}{2}(\kappa_n + \kappa_{n+1})]^2 \right. \\ &\qquad \left. - \frac{1}{\beta R}\log\cosh\left[\frac{\beta R}{2\tau}(\kappa_{n+1} - \kappa_n)(\xi - \chi_n)\right]\right\}_{\xi = \frac{1}{2}(\chi_{n-1}+\chi_n)}^{\xi = \xi_1}, \end{aligned}\right\} \tag{5.9}$$

for $\qquad \tfrac{1}{2}(\chi_{n-1} + \chi_n) < \xi_i < \tfrac{1}{2}(\chi_n + \chi_{n+1}).$

For extremely large Reynolds numbers and times such that $1 \ll \tau \ll R$, (5.9) is reduced to

$$\left.\begin{aligned} \phi_1 &= \phi_1(\tfrac{1}{2}(\chi_{n-1}+\chi_n)) - \frac{1}{4\tau}\frac{\gamma-3}{\gamma+1} \\ &\qquad \times \left\{[\xi-\tfrac{1}{2}(\kappa_n+\kappa_{n+1})]^2-(\kappa_{n+1}-\kappa_n)\,|\xi-\chi_n|\right\}_{\xi=\frac{1}{2}(\chi_{n-1}+\chi_n)}^{\xi=\xi_2}, \\ \phi_2 &= \phi_2(\tfrac{1}{2}(\chi_{n-1}+\chi_n)) + \frac{1}{4\tau}\frac{\gamma-3}{\gamma+1} \\ &\qquad \times \left\{[\xi-\tfrac{1}{2}(\kappa_n+\kappa_{n+1})]^2-(\kappa_{n+1}-\kappa_n)\,|\xi-\chi_n|\right\}_{\xi=\frac{1}{2}(\chi_{n-1}+\chi_n)}^{\xi=\xi_1}, \end{aligned}\right\} \quad (5.10)$$

for $\tfrac{1}{2}(\chi_{n-1}+\chi_n) < \xi_i < \tfrac{1}{2}(\chi_n+\chi_{n+1})$.

The order of magnitude of ϕ_i specified by (5.9) or (5.10) may be estimated as follows. The lengths $(\kappa_{n+1}-\kappa_n)$ and $[\lambda_n-\frac{1}{2}(\kappa_n+\kappa_{n+1})]$, which are all finite in the ξ_i co-ordinates, are $O(\epsilon)$, according to (3.10). On the other hand, the velocities v_{in} and V_{in} are $O(1)$, in view of the relation (4.5). Then, it follows from (5.5) and (5.6) that $\tau = O(\epsilon)$, so that $t = O(\epsilon^{-1})$, from (3.10). The above estimation of order of magnitude of terms in (5.9) or (5.10) shows that the ϕ_i are $O(\epsilon)$. Hence,

$$\frac{\partial \xi_i}{\partial x} = \epsilon\left(1-\frac{\partial \phi_i}{\partial x}\right) = \epsilon + O(\epsilon^2). \quad (5.11)$$

If we assume spatial periodicity of the turbulent field with length scale L, and require the vanishing space average of fluctuating quantities at an initial instant

$$\frac{1}{L}\int_0^L W^{(1)}(x,0)\,dx = 0,$$

then it follows from (4.1) that the F_i are also periodic, and

$$\frac{1}{L'}\int_0^{L'} F_i(\xi,0)\,d\xi = 0, \quad L' = \epsilon L, \quad (5.12)$$

where (5.11) has been taken into account, with $O(\epsilon^2)$ neglected. Since the F_i are solutions of (3.25), it is readily shown that the periodicity and the vanishing of the space average of F_i,

$$\frac{1}{L'}\int_0^{L'} F_i(\xi,\tau)\,d\xi = 0, \quad (5.13)$$

hold at all times $\tau > 0$.

The kinetic energy of turbulence per unit length is defined by

$$\mathscr{E}(t) = \lim_{L\to\infty}\frac{1}{2L}\int_0^L \rho(x,t)\,u(x,t)^2\,dx. \quad (5.14)$$

In the weak-shock approximation, (5.14) becomes

$$\begin{aligned} \mathscr{E}(t) &= \lim_{L\to\infty}\frac{\rho_0\epsilon^2}{2L}\int_0^L u^{(1)}(x,t)^2\,dx \\ &= \lim_{L'\to\infty}\frac{\rho_0\epsilon^2}{2L'}\left(\frac{2}{\gamma+1}\right)^2\int_0^{L'}[F_1(\xi,\tau)+F_2(\xi,\tau)]^2\,d\xi \\ &= \mathscr{E}_1(t)+\mathscr{E}_2(t), \end{aligned} \quad (5.15)$$

$$\mathscr{E}_i(t) = \lim_{L'\to\infty}\frac{2\rho_0\epsilon^2}{(\gamma+1)^2}\frac{1}{L'}\int_0^{L'} F_i(\xi,\tau)^2\,d\xi, \quad (5.16)$$

where the independence of the two components F_1 and F_2 has been assumed besides the condition (5.13). Since the energies $\mathscr{E}_1(t)$ and $\mathscr{E}_2(t)$ change in time according to the energy decay law of Burgers turbulence, the total kinetic energy $\mathscr{E}(t)$ must also decay in the same manner.

The same argument can be applied to correlations, energy spectra and other statistical quantities, so long as they are expressed in terms of integrals in space; and it may be concluded that the general statistical behaviour of weak-shock turbulence in a compressible fluid is identical with that of the Burgers model of turbulence.

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